Skripta riješenih zadataka

Kolegij: Otpornost materijala 1

Pripremili: Torić Neira
Šćulac Paulo
Škec Leo

Literatura:

- Šimić, V.: Otpornost materijala I, Školska knjiga, Zagreb, 2002.
- Šimić, V.: Otpornost materijala II, Školska knjiga, Zagreb, 2002.
- Brnić, J.,Turkalj,G.: Nauka o čvrstoći I, Tehnički fakultet Sveučilišta u Rijeci, 2004.
- Brnić, J.,Turkalj,G.: Nauka o čvrstoći II, Zigo, Rijeka, 2006.
- Alfirević, I.: Nauka o čvrstoći I, Tehnička knjiga, Zagreb, 1995.
- Bazjanac D.: Nauka o čvrstoći, Tehnička knjiga, Zagreb, 1973.
- Timošenko S.: Otpornost materijala 1, Građevinska knjiga, Beograd, 1972.
- Timošenko S.: Otpornost materijala 2, Građevinska knjiga, Beograd, 1966.
- Benham P.P., Crawford R.J.: Mechanics of Engineering Materials, Longman Scientific and Technical, Harlow, 1988.
- Beer F.P., Johnston E.R.: Mechanics of Materials, McGraw-Hill, London, 1992.
- Stanek M., Turk G.: Osnove mehanike trdnih teles, Fakulteta za gradbeništvo in geodezijo Univerze v Ljubljani, Ljubljana, 1996.
- Rašković D.: Otpornost materijala, Građevinska knjiga, Beograd, 1985.
- Brčić V.: Otpornost materijala, Građevinska knjiga, Beograd, 1982.

1 Zadat ak

Za konzolni nosač prikazan na slici potrebno je izračunati naprezanja i nacrtati dijagram naprezanja.

$F_1 = F_2 = 2 \cdot 10^4 \, N$

$F_3 = 4 \cdot 10^4 \, N$

$E = 2{,}2 \cdot 10^{11} \, Pa$

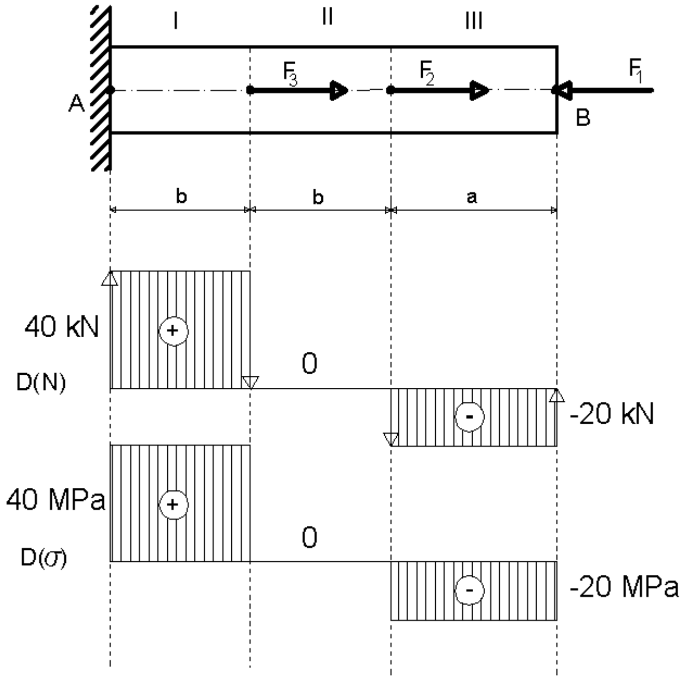
$A = 10 cm^2 = 10^{-3} m^2$

$a = 2m$

$b = 1m$

$1Pa = 1\frac{N}{m^2}$

$1MPa = 1\frac{N}{mm^2}$



Vrijednosti uzdužnih sila na pojedinim segmentima :

$N_{III} = -F_1 = -2 \cdot 10^4 \, N$

$N_{II} = -F_1 + F_2 = 0$

$N_I = F_3 + F_2 - F_1 = 4 \cdot 10^4 \, N$

Vrijednosti naprezanja na pojedinim segmentima :

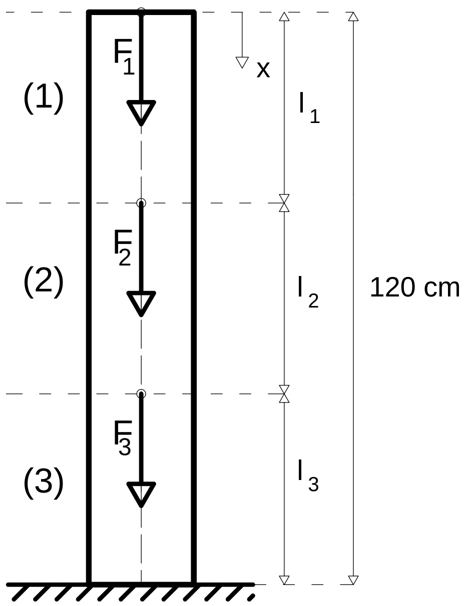
$$\sigma_{III} = \frac{N_{III}}{A} = -20 \cdot 10^6 \, Pa = -20 MPa$$

$$\sigma_I = \frac{N_{II}}{A} = 0$$

$$\sigma_I = \frac{N_I}{A} = 40 \cdot 10^6 \, Pa = 40 MPa$$

2. Zadat ak

Cilindrični stup promjera 4 cm, dužine 120 cm opterećen je na udaljenostima 0, 40 i 80 cm od slobodnog kraja aksijalnim silama F_1= 15 kN, F_2= 10 kN i F_3= 5 kN.
Izračunati naprezanja u pojedinim dijelovima stupa i pomak slobodnog kraja.



$E = 2 \cdot 10^5\, MPa$

$d = 4cm$

$L = 120cm; \qquad l_1 = l_2 = l_3 = l = \dfrac{L}{3} = 40cm$

$A = \dfrac{d^2\pi}{4} = \dfrac{4^2 \cdot 10^{-4}\pi}{4} = 12{,}56 \cdot 10^{-4}\, m^2$

Naprezanja po dijelovima:

$(1)K \;\; 0 \le x < 40cm$

$N_{(1)} = -F_1 = -15kN$

$$\sigma_{(1)} = \frac{N_{(1)}}{A} = \frac{-15 \cdot 10^3}{12{,}56 \cdot 10^{-4}} \frac{N}{m^2} = -12 \cdot 10^6\, Pa = -12MPa$$

$(2)K \;\; 40 \le x < 80cm$

$N_{(2)} = -F_1 - F_2 = -25kN$

$$\sigma_{(2)} = \frac{N_{(2)}}{A} = \frac{-25 \cdot 10^3}{12{,}56 \cdot 10^{-4}} \frac{N}{m^2} = -20 \cdot 10^6\, Pa = -20MPa$$

$(3)K \;\; 80 \le x < 120cm$

$N_{(3)} = -F_1 - F_2 - F_3 = -30kN$

$$\sigma_{(3)} = \frac{N_{(3)}}{A} = \frac{-30 \cdot 10^3}{12{,}56 \cdot 10^{-4}} \frac{N}{m^2} = -24 \cdot 10^6\, Pa = -24MPa$$

Pomak slobodnog kraja:

$$\Delta l = \Delta l_1 + \Delta l_2 + \Delta l_3 = \frac{l}{AE}\left(N_{(1)} + N_{(2)} + N_{(3)}\right) = \sum_{i=1}^{3} k \cdot N_{(i)}$$

$$k = \frac{l}{AE} = \frac{0{,}4}{12{,}56 \cdot 10^{-4} \cdot 2 \cdot 10^{11}} = 0{,}159 \cdot 10^{-8} \frac{m}{N}$$

$\Delta l_1 = k \cdot N_{(1)} = 0{,}159 \cdot 10^{-8} \cdot \left(-15 \cdot 10^3\right) = -2{,}415 \cdot 10^{-5} m = -0{,}024mm$

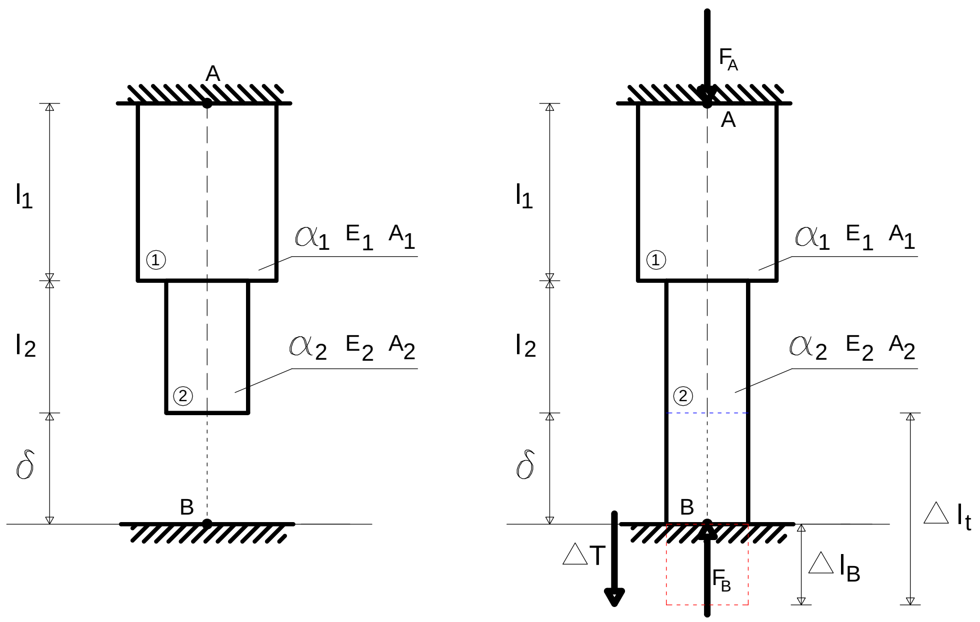
$\Delta l_2 = k \cdot N_{(2)} = 0{,}159 \cdot 10^{-8} \cdot \left(-25 \cdot 10^3\right) = -4{,}00 \cdot 10^{-5} m = -0{,}04mm$

$\Delta l_3 = k \cdot N_{(3)} = 0{,}159 \cdot 10^{-8} \cdot \left(-30 \cdot 10^3\right) = -4{,}80 \cdot 10^{-5} m = -0{,}048mm$

$\Delta l = \Delta l_1 + \Delta l_2 + \Delta l_3 = -0{,}112mm$

3. Zadatak

Za sastavljeni štap koji je kraći za δ od razmaka dviju krutih nepomičnih stijenki potrebno je odrediti naprezanja u štapu pri promjeni temperature za $+\Delta T$.



Ukoliko nema vanjskog otpora izduženju štapa, nema niti naprezanja u štapu.
To znači da se štap AC može slobodno izdužiti pod utjecajem temperature za veličinu δ.
Sve dok je $\Delta l_t \leq \delta$ u štapu nema naprezanja.
Ukoliko je tendencija štapa da se izduži za $\Delta l_t > \delta$, aktivira se u štapu sila koja opet " vraća " štap na realnu dužinu AB, kao što se vidi na slici. Može se zaključiti i da će sila u štapu biti tlačna, odnosno s predznakom – (minus).

δ – realno izduženje

Iz uvjeta ravnoteže štapa: $F_A - F_B = 0 \quad \rightarrow F_A = F_B$

$$\Delta l_t = \alpha_1 l_1 \Delta T + \alpha_2 l_2 \Delta T$$
$$\Delta l_t \leq \delta \quad \rightarrow \varepsilon > 0, \quad \sigma = 0$$
$$\Delta l_t > \delta \quad \rightarrow \varepsilon > 0, \quad \sigma \neq 0$$

$$\delta = \Delta l_t - \Delta l_B \ldots (1)$$
$$\Delta l_t = (\alpha_1 l_1 + \alpha_2 l_2)\Delta T \ldots (2)$$
$$\Delta l_B = \frac{F_B l_1}{E_1 A_1} + \frac{F_B l_2}{E_2 A_2} \ldots (3)$$

$$(2);(3) \rightarrow (1) \Rightarrow F_B = \frac{[(\alpha_1 l_1 + \alpha_2 l_2)\Delta T - \delta]E_1 A_1}{l_1\left(1 + \frac{l_2}{l_1}\frac{E_1 A_1}{E_2 A_2}\right)}$$

$$\sigma_{x1} = -\frac{F_B}{A_1} = -\frac{[(\alpha_1 l_1 + \alpha_2 l_2)\Delta T - \delta]E_1}{l_1\left(1 + \frac{l_2}{l_1}\frac{E_1 A_1}{E_2 A_2}\right)}$$

$$\sigma_{x2} = -\frac{F_B}{A_2} = -\frac{[(\alpha_1 l_1 + \alpha_2 l_2)\Delta T - \delta]E_1 A_1}{A_2 l_1\left(1 + \frac{l_2}{l_1}\frac{E_1 A_1}{E_2 A_2}\right)}$$

4. Zadatak

Dimenzionirati štapove AB i DG kružnog poprečnog presjeka napravljene od čelika, te odrediti njihova produljenja.

$\sigma_{dop} = 140 Mpa$

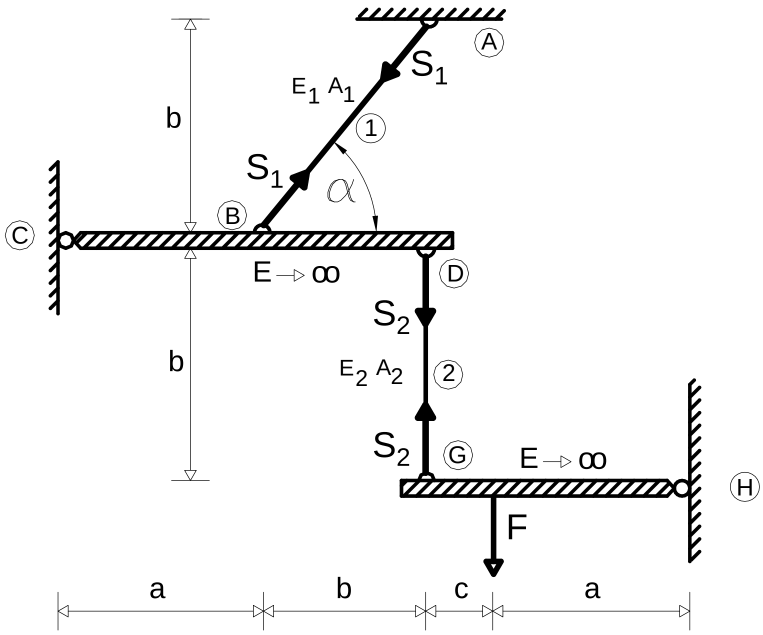
$E = 2 \cdot 10^5 Mpa$

$F = 100 kN$

$a = 3m$

$b = 2m$

$c = 1m$



$l_1 = \sqrt{2^2 + 2^2} = 2\sqrt{2} m$

$l_2 = 2m$

Iz sume momenata oko točke H i C dobivamo sile u štapovima 1 i 2.

$$\underline{\sum M_H = 0} \Rightarrow S_2 \cdot 4 - F \cdot 3 = 0 \rightarrow S_2 = \frac{3}{4} F = \frac{3}{4} 100 = 75 kN$$

$$\underline{\sum M_C = 0} \Rightarrow S_2 \cdot 5 - S_1 \cdot \sin 45° \cdot 3 = 0 \rightarrow S_1 = S_2 \frac{5}{3 \cdot \sin 45°} = 75 \frac{5}{3 \cdot \sin 45°} = 176,8 kN$$

Dimenzioniranje:

$$\sigma_x = \frac{S}{A} \leq \sigma_{dop} \Rightarrow A_{pot} \geq \frac{S}{\sigma_{dop}}$$

$$A_{1pot} \geq \frac{S_1}{\sigma_{dop}} = \frac{176,8 \cdot 10^3}{140 \cdot 10^6} = 1,26 \cdot 10^{-3} m^2 = 12,6 cm^2$$

$$A_1 = \frac{d_1^{\,2} \pi}{4} \Rightarrow d_1 = \sqrt{\frac{4A_1}{\pi}} = 4,01 cm \rightarrow usvojeno \rightarrow d_1 = 42 mm \rightarrow (A_1 = 13,85 cm^2)$$

$$A_{2pot} \geq \frac{S_2}{\sigma_{dop}} = \frac{75 \cdot 10^3}{140 \cdot 10^6} = 0,536 \cdot 10^{-3} m^2 = 5,36 cm^2$$

$$A_2 = \frac{d_2^{\,2} \pi}{4} \Rightarrow d_2 = \sqrt{\frac{4A_2}{\pi}} = 2,61 cm \rightarrow usvojeno \rightarrow d_1 = 28 mm \rightarrow (A_2 = 6,15 cm^2)$$

Kontrola

$$\sigma_{x1} = \frac{S_1}{A_1} = \frac{176{,}8 \cdot 10^3}{13{,}85 \cdot 10^{-4}} = 127{,}6 MPa < \sigma_{dop} = 140 MPa$$

$$\sigma_{x2} = \frac{S_2}{A_2} = \frac{75 \cdot 10^3}{6{,}15 \cdot 10^{-4}} = 121{,}9 MPa < \sigma_{dop} = 140 MPa$$

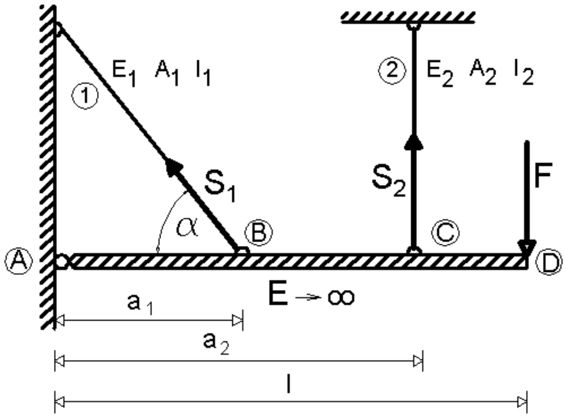
Produljenja štapova

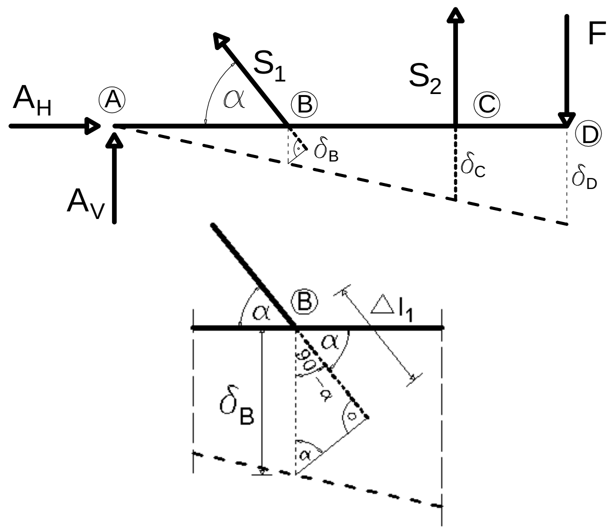
$$\Delta l_1 = \frac{S_1 l_1}{E_1 A_1} = \frac{176{,}8 \cdot 10^3 \cdot 2\sqrt{2}}{2 \cdot 10^{11} \cdot 13{,}85 \cdot 10^{-4}} = 18{,}056 \cdot 10^{-4} m = 0{,}18 cm$$

$$\Delta l_2 = \frac{S_2 l_2}{E_2 A_2} = \frac{75 \cdot 10^3 \cdot 2}{2 \cdot 10^{11} \cdot 6{,}15 \cdot 10^{-4}} = 12{,}195 \cdot 10^{-4} m = 0{,}12 cm$$

5. Zadat ak

Odrediti naprezanja u štapovima 1 i 2 te vertikalni pomak točke D.





Sustav je jedanput statički neodređen (ima jednu prekomjernu veličinu), odnosno nepoznate su 4 veličine (A_H , A_V , S_1 , S_2).
Uz tri uvjeta ravnoteže postavljamo dodatnu jednadžbu na deformiranom sustavu na principu sličnosti trokuta.

Iz uvjeta ravnoteže sila :

$$\underline{\Sigma M_A = 0} \rightarrow Fl - S_2 a_2 - S_1 a_1 \sin\alpha = 0 \ldots (1)$$

$$\underline{\Sigma X = 0} \rightarrow A_H - S_1 \cos\alpha = 0 \rightarrow A_H = S_1 \cos\alpha \ldots (2)$$

$$\underline{\Sigma Y = 0} \rightarrow S_2 - F + S_1 \sin\alpha + R_V = 0 \ldots (3)$$

Iz plana pomaka :

$$\delta_C \equiv \Delta l_2; \qquad \delta_B = \frac{\Delta l_1}{\sin\alpha} \quad \ldots (4)$$

Iz sličnosti trokuta :

$$\frac{\delta_C}{a_2} = \frac{\delta_B}{a_1} \ldots (5)$$

Iz Hookovog zakona :

$$\Delta l_1 = \frac{S_1 l_1}{E_1 A_1}; \quad \Delta l_2 = \frac{S_2 l_2}{E_2 A_2} \quad \ldots (6)$$

$$(4) \rightarrow (5) \qquad \frac{\Delta l_2}{a_2} = \frac{\Delta l_1}{a_1 \sin\alpha} \quad \ldots (7)$$

$(6) \rightarrow (7)$

$$\frac{S_2 l_2}{E_2 A_2 a_2} = \frac{S_1 l_1}{E_1 A_1 a_1 \sin\alpha}$$

$$S_1 = S_2 \frac{E_1 A_1}{E_2 A_2} \frac{l_2}{l_1} \frac{a_1}{a_2} \sin\alpha$$

$$S_1 \rightarrow (1) \Rightarrow S_2 = \frac{Fl}{a_2\left[1 + \frac{E_1 A_1}{E_2 A_2} \frac{l_2}{l_1} \frac{a_1^{\,2}}{a_2^{\,2}} \sin^2\alpha\right]}$$

Naprezanja u štapovima:

$$\sigma_1 = \frac{S_1}{A_1}$$

$$\sigma_2 = \frac{S_2}{A_2}$$

Vertikalni pomak točke D ($\delta_C \equiv \Delta l_2$):

$$\frac{\delta_C}{a_2} = \frac{\delta_D}{l} \Rightarrow \quad \delta_D = \Delta l_2 \frac{l}{a_2} = \frac{S_2 l_2 l}{E_2 A_2 a_2}$$

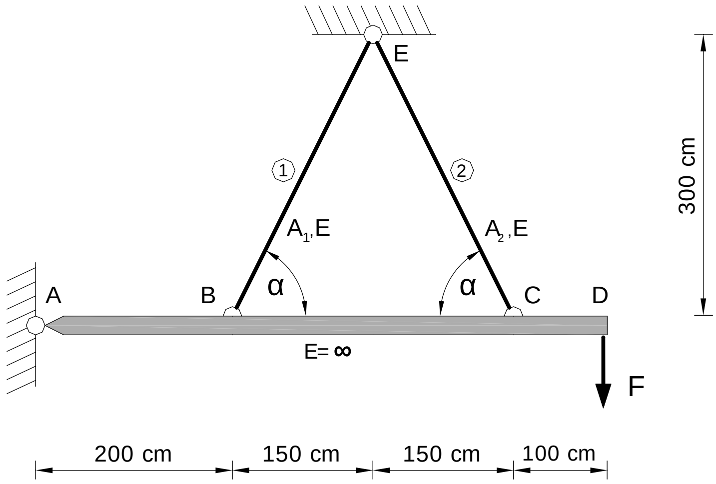
6. Zadatak

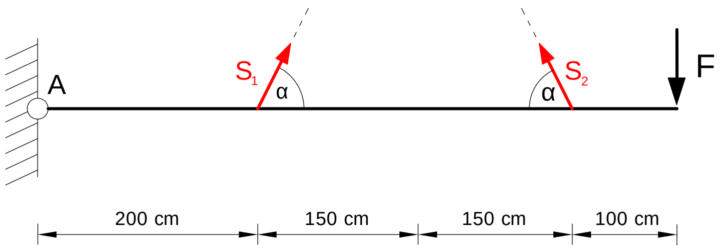
ABCD, beskonačno velike krutosti, zglobno je povezana u točki A i ovješena o dva čelična štapa BE i CE te opterećena silom F.
Potrebno je odrediti dopuštenu veličinu sile F iz uvjeta da naprezanja u štapovima BE i CE ne prekorače dopuštenu vrijednost naprezanja $\sigma_{dop} = 140$ MPa.

$E = 2{,}0 \cdot 10^5$ N/mm^2
$A_1 = 4{,}0$ cm^2
$A_2 = 1{,}5\ A_1 = 6{,}0$ cm^2

$$tg\alpha = \frac{3}{1{,}5} = 2{,}0 \quad \rightarrow \alpha = 63{,}43°$$

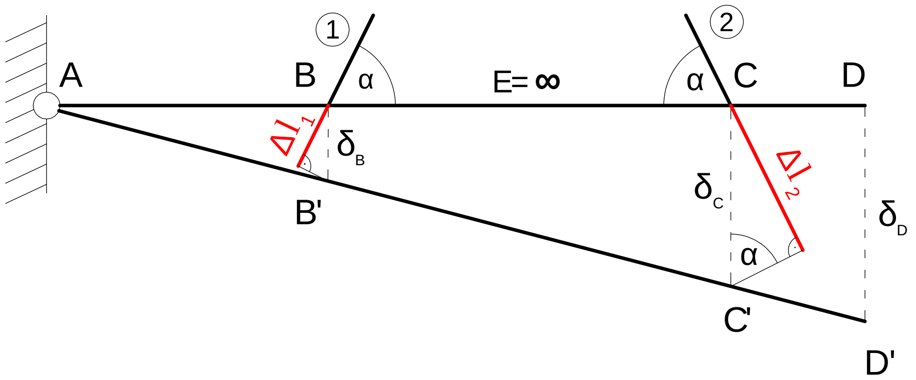
$$l_1 = l_2 = \sqrt{3^2 + 1{,}5^2} = 3{,}354\ m$$





$$\sum M_A = 0$$

$$S_1 \sin\alpha \cdot 2 + S_2 \sin\alpha \cdot 5 - F \cdot 6 = 0 \quad \text{K } (1)$$



Iz plana pomaka dobivamo sljedeće podatke:

$$\frac{\delta_B}{2} = \frac{\delta_C}{5} \quad \rightarrow \quad \delta_B = \frac{2}{5}\delta_C \quad \text{K } (2)$$

$$sin\alpha = \frac{\Delta l_1}{\delta_B} \quad \rightarrow \quad \delta_B = \frac{\Delta l_1}{sin\alpha} \quad \text{K } (3)$$

$$sin\alpha = \frac{\Delta l_2}{\delta_C} \quad \rightarrow \quad \delta_C = \frac{\Delta l_2}{sin\alpha} \quad \text{K } (4)$$

Uvrštavanjem jednadžbi (3) i (4) $\rightarrow$ (2) uz $\Delta l_1 = \dfrac{S_1 l_1}{EA_1}$ i $\Delta l_2 = \dfrac{S_2 l_2}{EA_2} = \dfrac{S_2 l_2}{E \cdot \frac{3}{2} \cdot A_1} = \dfrac{2}{3}\dfrac{S_2 l_2}{EA_1}$ slijedi:

$$\frac{\Delta l_1}{sin\alpha} = \frac{2}{5}\frac{\Delta l_2}{sin\alpha} \quad / \cdot sin\alpha$$

$$\Delta l_1 = \frac{2}{5}\Delta l_2$$

$$\frac{S_1 l_1}{EA_1} = \frac{2}{5} \cdot \frac{2}{3}\frac{S_2 l_2}{EA_1} \quad / \cdot EA_1$$

$$S_1 l_1 = \frac{4}{15} S_2 l_2 \qquad (l_1 = l_2) \rightarrow S_1 = \frac{4}{15} S_2 \quad \text{K } (5)$$

(5) $\rightarrow$ (1)

$$\frac{4}{15} \cdot 2 sin\alpha \cdot S_2 + 5 sin\alpha \cdot S_2 = 6F$$

$$S_2\left(\frac{8}{15} sin\alpha + 5 sin\alpha\right) = 6F \quad \rightarrow \quad S_2 = \frac{6F}{\frac{83}{15} sin\alpha} = \frac{90}{83 sin\alpha} F = 1{,}2124F$$

Uvrstimo li dobivenu vrijednost sile u štapu 2 S_2 u (5) dobivamo S_1 u ovisnosti o sili F:

$$S_1 = \frac{4}{15}\frac{90}{83 sin\alpha} F = 0{,}3233F \quad .$$

Dopuštenu veličinu sile F dobivamo iz uvjeta da naprezanja u štapovima ne prekorače dopuštene vrijednosti naprezanja:

$$\sigma_1 = \frac{S_1}{A_1} \le \sigma_{dop}$$

$$\frac{0{,}3233F}{A_1} \le \sigma_{dop} \quad \rightarrow \quad F_{dop} \le \frac{\sigma_{dop} \cdot A_1}{0{,}3233} = \frac{140 \frac{N}{mm^2} \cdot 400 mm^2}{0{,}3233}$$

$$F_{dop} \le 173214\, N = 173{,}214\, kN$$

$$\sigma_2 = \frac{S_2}{A_2} \le \sigma_{dop}$$

$$\frac{1{,}2124F}{A_2} \le \sigma_{dop} \quad \rightarrow \quad F_{dop} \le \frac{\sigma_{dop} \cdot A_2}{1{,}2124} = \frac{140 \frac{N}{mm^2} \cdot 600 mm^2}{1{,}2124}$$

$$F_{dop} \le 69284\, N = 69{,}284\, kN$$

Mjerodavno je $F_{dop} \le 69284\, N = 69{,}284\, kN$.

7. Zadat ak

Dimenzionirati čeličnu zategu kružnog porečnog presjeka (A_1) i drveni kosnik pravokutnog poprečnog presjeka (h=2b) površine A_2 = $10A_1$ ako je F=155 kN a dopuštena naprezanja za čelik $12 \cdot 10^7$ N/m^2 i drvo $6 \cdot 10^6$ N/m^2.

$$\sigma_{1dop} = 12 \cdot 10^7 \frac{N}{m^2}$$

$$\sigma_{2dop} = 6 \cdot 10^6 \frac{N}{m^2}$$

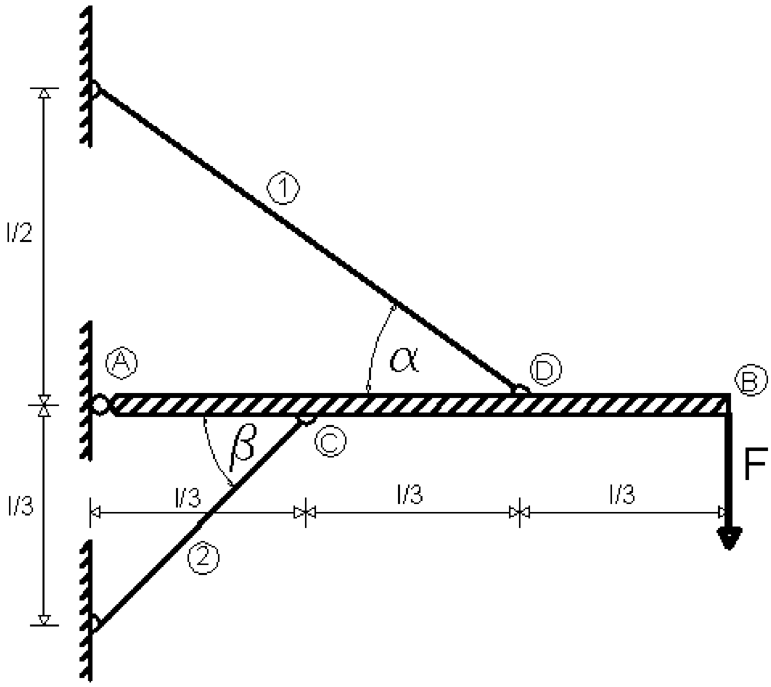
$$E_1 = 2 \cdot 10^{11} \frac{N}{m^2}$$

$$E_2 = 1 \cdot 10^{10} \frac{N}{m^2}$$

$$E_1 = 20 E_2$$

$$A_2 = 10 A_1$$

$$h = 2b$$



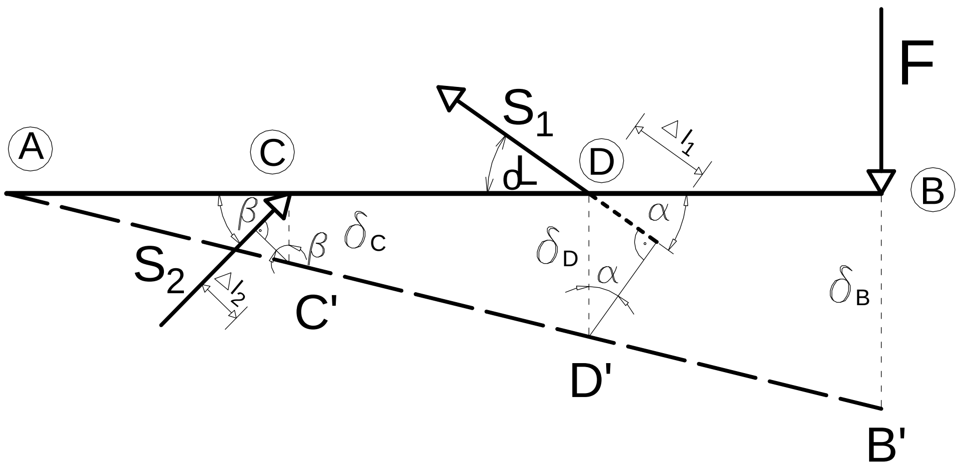
$$l_1 = \sqrt{\left(\frac{l}{2}\right)^2 + \left(\frac{2l}{3}\right)^2} = \sqrt{\frac{l^2}{4} + \frac{4l^2}{9}} = \sqrt{\frac{25l^2}{36}} = \frac{5}{6} l$$

$$l_2 = \sqrt{2 \cdot \left(\frac{l}{3}\right)^2} = \frac{\sqrt{2}}{3} l$$

$$\sin\alpha = \frac{\frac{l}{2}}{\frac{5l}{6}} = \frac{3}{5} \qquad \cos\alpha = \frac{\frac{2l}{3}}{\frac{5l}{6}} = \frac{4}{5}$$

$$\sin\beta = \cos\beta = \frac{\sqrt{2}}{2}$$

Plan pomaka :



Sile u štapovima :

$$\underline{\sum M_A = 0} \quad \rightarrow Fl - S_1 \sin\alpha \frac{2}{3} l - S_2 \sin\beta \frac{1}{3} l = 0 \ldots (1)$$

$$\frac{\delta_c}{\frac{1}{3}l} = \frac{\delta_D}{\frac{2}{3}l} \quad \rightarrow \delta_D = 2\delta_C \ldots (2)$$

$$\delta_C = \frac{\Delta l_2}{\sin\beta} = \frac{S_2 l_2}{E_2 A_2 \sin\beta}; \quad \delta_D = \frac{\Delta l_1}{\sin\alpha} = \frac{S_1 l_1}{E_1 A_1 \sin\alpha}$$

$$(2) \Rightarrow \frac{S_1 l_1}{E_1 A_1 \sin\alpha} = 2 \frac{S_2 l_2}{E_2 A_2 \sin\beta} \quad \rightarrow S_1 = 2 \frac{E_1 A_1 l_2 \sin\alpha}{E_2 A_2 l_1 \sin\beta} S_2$$

$$S_1 = 2 \frac{20E_2 \cdot A_1 \cdot 5l \cdot 3 \cdot 3 \cdot 2}{E_2 \cdot 10A_1 \cdot 6 \cdot \sqrt{2}l \cdot 5 \cdot \sqrt{2}} S_2 \Rightarrow S_1 = 6S_2 \rightarrow (1) \Rightarrow$$

$$\Rightarrow S_2 = \frac{F}{\frac{1}{3}\sin\beta + 4\sin\alpha} = \frac{F}{\left(\frac{12}{5} + \frac{1}{6}\sqrt{2}\right)} = 0{,}38F \rightarrow \underline{S_2 = 58{,}9kN}; \underline{S_1 = 353{,}4kN}$$

Dimenzioniranje :

$$\sigma_1 = \frac{S_1}{A_1} \le \sigma_{1dop} \rightarrow A_1 = \frac{d^2\pi}{4} \ge \frac{S_1}{\sigma_{1dop}} \rightarrow d \ge \sqrt{\frac{4S_1}{\pi\sigma_{1dop}}} = \sqrt{\frac{4 \cdot 353{,}4 \cdot 10^3}{\pi \cdot 12 \cdot 10^7}}$$

$$d \ge 0{,}06m = 6cm \rightarrow \underline{d = 7cm}$$

$$A_2 = 10A_1 = 10 \cdot 38{,}48 = 384{,}8cm^2$$

$$A_2 = b \cdot h = 2b^2 = 384{,}8cm^2 \Rightarrow b = \sqrt{\frac{A_2}{2}} = \sqrt{\frac{384{,}8}{2}} = 13{,}87cm$$

$$\underline{b = 13{,}87cm}$$

$$\underline{h = 27{,}74cm}$$

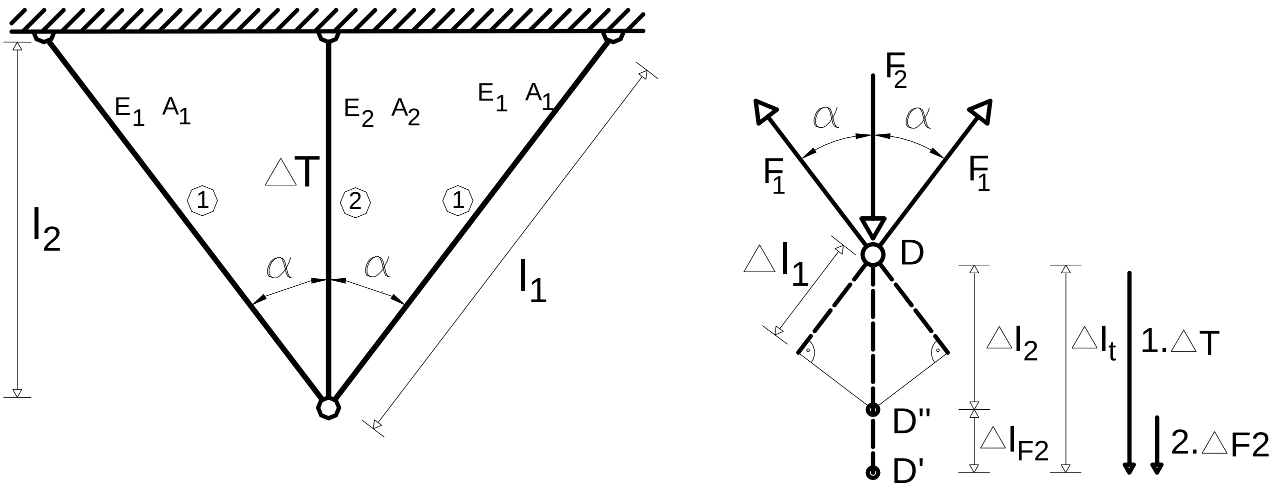
Kontrola naprezanja :

$$\sigma_1 = \frac{S_1}{A_1} = \frac{353{,}4 \cdot 10^3}{\frac{7^2 \cdot 10^{-4}\pi}{4}} = 91{,}84MPa \le \sigma_{1dop} = 120MPa$$

$$\sigma_2 = \frac{S_2}{A_2} = \frac{58{,}9 \cdot 10^3}{2 \cdot 13{,}87^2 \cdot 10^{-4}} = 1{,}53MPa \le \sigma_{2dop} = 6MPa$$

8. Zadatak

Odrediti naprezanja u štapovima ukoliko se temperatura štapa 2 poveća za ΔT :



Iz plana pomaka i opterećenja (ovdje promjena temperature) možemo zaključiti da se štap 2 nastoji izdužiti, no njegovo potpuno izduženje sprječavaju druga dva štapa spojena u čvoru D, koji nisu direktno opterećeni i nemaju tendenciju izduženja.
Štap 2 se izduži za Δl_t pri čemu čvor D zauzme položaj D'. Tada se javi otpor štapova 1 koji u štapu 2 aktivira tlačnu silu F_2 koja čvor D vraća iz D' u D''.
Budući da su sva tri štapa međusobno spojena u čvoru D, izduženjem štapa 2 prisilno se izdužuju i štapovi s brojem 1. Kao posljedica izduženja u njima se javlja vlačna sila F_1 (vlačna sila uzrokuje izduženje ako nema nikakvog drugog opterećenja).

Iz uvjeta ravnoteže sila :

$$\underline{\Sigma Y = 0} \quad \rightarrow 2F_1 \cos\alpha - F_2 = 0 \quad 2F_1 \cos\alpha = F_2 \ldots (1)$$

Iz plana pomaka :

$$\cos\alpha = \frac{l_2}{l_1} = \frac{\Delta l_1}{\Delta l_2} \rightarrow \quad \Delta l_1 = \Delta l_2 \cos\alpha \ldots (2)$$

Iz Hookovog zakona :

$$\Delta l_1 = \frac{F_1 l_1}{E_1 A_1} \ldots (3) \qquad \Delta l_2 = \Delta l_t - \Delta l_{F2} = \alpha_2 \cdot \Delta T \cdot l_2 - \frac{F_2 l_2}{E_2 A_2} \ldots (4)$$

Ukupno izduženje Δl_2 = utjecaj temp.(=povećanje) + utjecaj tlačne sile(=skraćenje: predznak –)

$$(3),(4) \rightarrow (2) \quad \frac{F_1 l_1}{E_1 A_1} = (\alpha_2 \cdot \Delta T \cdot l_2 - \frac{F_2 l_2}{E_2 A_2}) \cos\alpha \ldots (5)$$

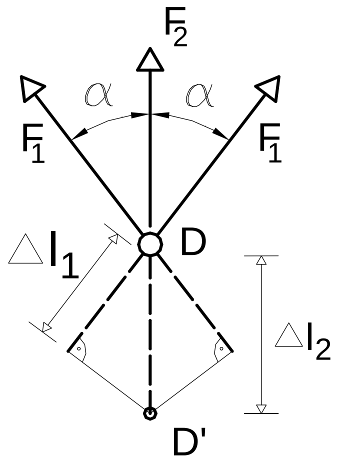
$$(1) \rightarrow (5) \quad \frac{F_1 l_1}{E_1 A_1 \cos\alpha} = \alpha_2 \cdot \Delta T \cdot l_2 - \frac{2F_1 l_2 \cos\alpha}{E_2 A_2}$$

$$l_2 = l_1 \cos\alpha \quad \rightarrow \quad F_1 = \frac{\alpha_2 \Delta T E_1 A_1 \cos^2\alpha}{1 + 2\dfrac{E_1 A_1}{E_2 A_2}\cos^3\alpha}$$

$$(1) \rightarrow F_2 = 2F_1 \cos\alpha \quad \Rightarrow F_2 = \frac{2\alpha_2 \Delta T E_1 A_1 \cos^3\alpha}{1 + 2\dfrac{E_1 A_1}{E_2 A_2}\cos^3\alpha}$$

Naprezanja u štapovima: $\sigma_1 = \dfrac{F_1}{A_1}; \quad \sigma_2 = -\dfrac{F_2}{A_2}$

Zadatak se može riješiti i na način da se u svim štapovima pretpostave vlačne sile (svi će se štapovi produžiti).



Iz uvjeta ravnoteže sila :

$\underline{\Sigma Y = 0} \quad \rightarrow 2F_1 \cos\alpha + F_2 = 0 \quad -2F_1 \cos\alpha = F_2 \ldots (1)$

Iz plana pomaka :

$\cos\alpha = \dfrac{l_2}{l_1} = \dfrac{\Delta l_1}{\Delta l_2} \rightarrow \quad \Delta l_1 = \Delta l_2 \cos\alpha \ldots (2)$

Iz Hookovog zakona :

$$\Delta l_1 = \frac{F_1 l_1}{E_1 A_1} \ldots (3) \qquad \Delta l_2 = \frac{F_2 l_2}{E_2 A_2} + \alpha_2 \cdot \Delta T \cdot l_2 \ldots (4)$$

Ukupno izduženje Δl_2 = utjecaj vlačne sile(=povećanje) + utjecaj temp.(=povećanje)

$$(3),(4) \rightarrow (2) \qquad \frac{F_1 l_1}{E_1 A_1} = \left(\frac{F_2 l_2}{E_2 A_2} + \alpha_2 \cdot \Delta T \cdot l_2\right)\cos\alpha \;...(5)$$

$$(1) \rightarrow (5) \qquad \frac{F_1 l_1}{E_1 A_1 \cos\alpha} = -\frac{2F_1 l_2 \cos\alpha}{E_2 A_2} + \alpha_2 \cdot \Delta T \cdot l_2$$

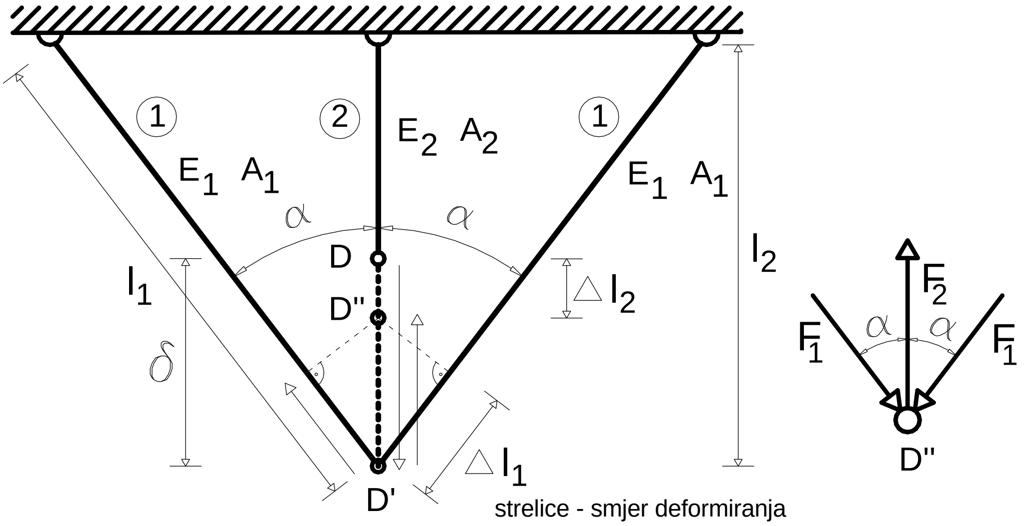
$$l_2 = l_1 \cos\alpha \quad \rightarrow \quad F_1 = \frac{\alpha_2 \Delta T E_1 A_1 \cos^2\alpha}{1 + 2\dfrac{E_1 A_1}{E_2 A_2}\cos^3\alpha}$$

$$(1) \rightarrow F_2 = -2F_1 \cos\alpha \quad \rightarrow F_2 = -\frac{2\alpha_2 \Delta T E_1 A_1 \cos^3\alpha}{1 + 2\dfrac{E_1 A_1}{E_2 A_2}\cos^3\alpha}$$

Naprezanja u štapovima: $\sigma_1 = \dfrac{F_1}{A_1}; \quad \sigma_2 = \dfrac{F_2}{A_2}$

9. Zadatak

Odrediti naprezanja u štapovima ako je srednji štap izveden kraći za δ od predviđene duljine l.



Iz uvjeta ravnoteže sila :

$$\underline{\Sigma Y = 0} \rightarrow F_2 - 2F_1 \cos\alpha = 0 \rightarrow F_2 = 2F_1 \cos\alpha \ldots (1)$$

Iz plana pomaka :

$$\Delta l_1 = \frac{F_1 l_1}{E_1 A_1}$$

$$\Delta l_2 = \frac{F_2 (l_2 - \delta)}{E_2 A_2}$$

$$\cos\alpha = \frac{l_2}{l_1} = \frac{\Delta l_1}{\delta - \Delta l_2}$$

$$\delta = \Delta l_2 + \frac{\Delta l_1}{\cos\alpha} = \frac{F_2 (l_2 - \delta)}{E_2 A_2} + \frac{F_1 l_1}{E_1 A_1}\frac{1}{\cos\alpha} \ldots (2)$$

$$(1) \rightarrow (2) \Rightarrow \delta = 2\frac{F_1 (l_2 - \delta)}{E_2 A_2}\cos\alpha + \frac{F_1 l_1}{E_1 A_1}\frac{1}{\cos\alpha} = F_1\left(2\frac{l_2 - \delta}{E_2 A_2}\cos\alpha + \frac{l_1}{E_1 A_1}\frac{1}{\cos\alpha}\right)$$

$$F_1 = \frac{E_1 A_1 E_2 A_2 \cos\alpha}{2E_1 A_1 (l_2 - \delta)\cos^2\alpha + E_2 A_2 l_1}\delta = \frac{E_1 A_1 E_2 A_2 \cos^2\alpha}{2E_1 A_1 (l_2 - \delta)\cos^3\alpha + E_2 A_2 l_2}\delta$$

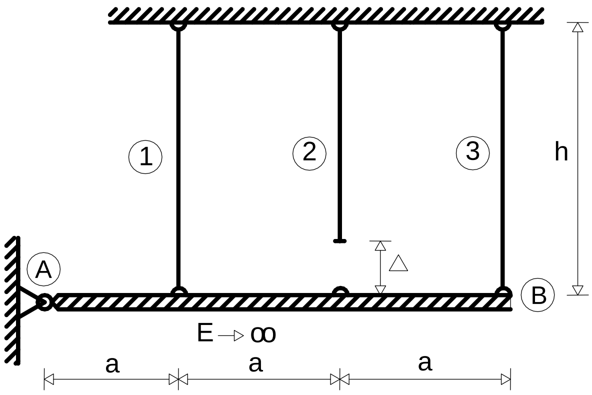
$$(1) \Rightarrow F_2 = \frac{2E_1 A_1 E_2 A_2 \cos^2\alpha}{2E_1 A_1 (l_2 - \delta)\cos^2\alpha + E_2 A_2 l_1}\delta = \frac{2E_1 A_1 E_2 A_2 \cos^3\alpha}{2E_1 A_1 (l_2 - \delta)\cos^3\alpha + E_2 A_2 l_2}\delta$$

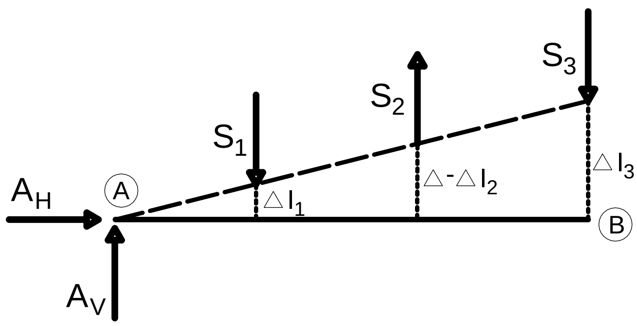
Naprezamja u štapovima $\sigma_1 = -\frac{F_1}{A_1}$ $\sigma_2 = \frac{F_2}{A_2}$

10. Zadatak

Kruta greda AB obješena je o tri čelična štapa istog poprečnog presjeka površine 10 cm^2. Dužina štapova je h = 1 m, s tim da je srednji štap (2) napravljen kraći od projektirane dužine za $\Delta = 0{,}6$ mm. Odrediti sile u štapovima i izduženje srednjeg štapa ako je izvršena prinudna montaža sustava.

$$E_1A_1 = E_2A_2 = E_3A_3 = EA$$
$$l_1 = l_2 = l_3 = h$$





$$\Delta l_1 = \frac{S_1 h}{EA}; \quad \Delta l_2 = \frac{S_2(h-\Delta)}{EA}; \quad \Delta l_3 = \frac{S_3 h}{EA}$$

$$\frac{\Delta l_3}{3a} = \frac{\Delta l_1}{a} \Rightarrow \Delta l_3 = 3\Delta l_1 \Rightarrow \frac{S_3 h}{EA} = 3\frac{S_1 h}{EA} \Rightarrow S_3 = 3S_1 \ldots (1)$$

$$\frac{\Delta l_1}{a} = \frac{\Delta - \Delta l_2}{2a} \Rightarrow 2\Delta l_1 = \Delta - \Delta l_2 \Rightarrow \frac{S_2(h-\Delta)}{EA} = \Delta - 2\frac{S_1 h}{EA} \Rightarrow S_2 = \frac{EA}{h-\Delta}\Delta - 2\frac{h}{h-\Delta}S_1 \ldots (2)$$

$$\underline{\Sigma M_A = 0} \rightarrow S_1 a - S_2 2a + S_3 3a = 0 \Rightarrow S_1 = 2S_2 - 3S_3 \ldots (3)$$

$$(1),(2) \rightarrow (3) \Rightarrow S_1 = 2\left(\frac{EA\Delta}{h-\Delta} - \frac{2h}{h-\Delta}S_1\right) - 9S_1 \rightarrow S_1 = \frac{2EA}{14h - 10\Delta}\Delta = 18kN$$

$$S_1 \rightarrow (1) \Rightarrow S_3 = 3S_1 = 54kN$$

$$S_1 \rightarrow (2) \Rightarrow S_2 = \frac{EA}{h-\Delta}\Delta - \frac{2h}{h-\Delta}S_1 = 90{,}05kN$$

$$\underline{S_1 = 18kN}$$

$$\underline{S_2 = 90{,}05kN}$$

$$\underline{S_3 = 54kN}$$

$$\Delta l_2 = \frac{S_2(h-\Delta)}{EA} = \frac{90 \cdot 10^3 \cdot (1 - 0{,}6 \cdot 10^{-3})}{2{,}1 \cdot 10^{11} \cdot 1 \cdot 10^{-3}} = 4{,}29 \cdot 10^{-4}\, m = 0{,}429\, mm$$

11. Zadatak

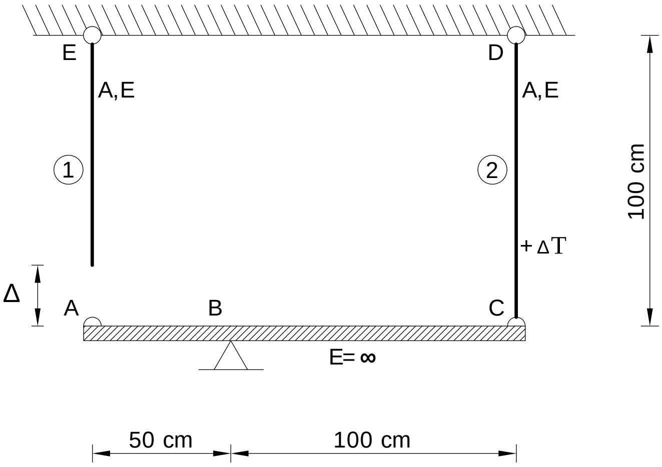
Greda ABC, beskonačno velike krutosti, ovješena je o 2 čelična štapa kružnog poprečnog presjeka promjera *d*. Pri izradi konstrukcije štap AE izrađen je kraći za Δ od potrebne duljine, te se pri njegovoj montaži morala upotrijebiti sila.

Potrebno je odrediti:

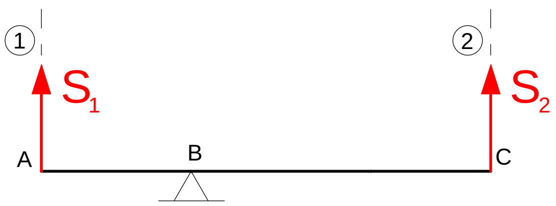
a) naprezanja u štapovima AE i CD ako se temperatura štapa CD poveća za $\Delta T = 25\,\mathrm{K}$,

b) vertikalni pomak točke C.

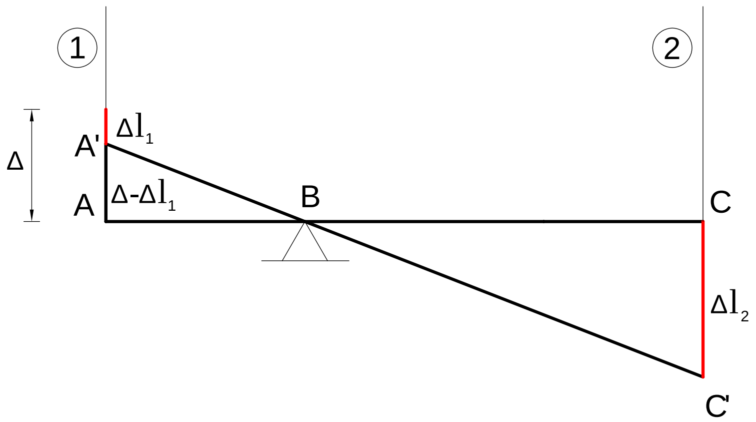
$$\Delta = 1\,mm$$
$$E = 2{,}1 \cdot 10^5\ \mathrm{MPa}$$
$$\alpha_T = 1{,}0 \cdot 10^{-5}\ \mathrm{K}^{-1}$$
$$d = 2\ \mathrm{cm}$$



$$l_1 = 1000mm - 1mm = 999\ mm$$
$$l_2 = 1{,}0\ m$$
$$A = \frac{d^2\pi}{4} = \frac{2^2\pi}{4} = 3{,}14\ cm^2$$



$$\sum M_B = 0$$
$$S_1 \cdot 0{,}5 - S_2 \cdot 1{,}0 = 0$$
$$S_1 = 2S_2 \quad \ldots(1)$$



$$\frac{\Delta l_2}{1{,}0} = \frac{\Delta - \Delta l_1}{0{,}5}$$
$$\Delta l_2 = 2(\Delta - \Delta l_1) \quad \ldots(2)$$

Produljenja štapova 1 i 2 definiramo kao:

$$\Delta l_1 = \frac{S_1 l_1}{EA} \quad \text{i} \quad \Delta l_2 = \frac{S_2 l_2}{EA} + \alpha_T \Delta T\, l_2$$

Uvrštavanjem produljenja Δl_1 i Δl_2 u (2) dobivamo:

$$\frac{S_2 l_2}{EA} + \alpha_T \Delta T\, l_2 = 2\left(\Delta - \frac{S_1 l_1}{EA} \right) \quad \mathrm{K}\ (3)$$

Uvrštavanjem (1) $\rightarrow$ (3) slijedi

$$\frac{S_2 l_2}{EA} + \alpha_T \Delta T\, l_2 = 2\Delta - 2\frac{2 S_2 l_1}{EA}$$

$$S_2\left(\frac{l_2}{EA} + 4\frac{l_1}{EA} \right) = 2\Delta - \alpha_T \Delta T\, l_2$$

$$S_2 = \frac{EA}{l_2 + 4 l_1}\left(2\Delta - \alpha_T \Delta T\, l_2\right) = \frac{2{,}1 \cdot 10^5\ \frac{N}{mm^2} \cdot 314 mm^2}{1000 mm + 4 \cdot 999 mm}\left(2 \cdot 1 mm - 1 \cdot 10^{-5} K^{-1} \cdot 25 K \cdot 1000 mm\right)$$

$$S_2 = 23097{,}5 N = 23{,}098\, kN$$

$$S_1 = 2 S_2 = 46195 N = 46{,}196\, kN$$

Naprezanja u štapovima iznose:

$$\sigma_1 = \frac{S_1}{A_1} = \frac{46196 N}{314 mm^2} = 147{,}12\, MPa$$

$$\sigma_2 = \frac{S_2}{A_2} = \frac{23098 N}{314 mm^2} = 73{,}56\, MPa$$

Vertikalni pomak točke C:

$$\Delta l_2 = \frac{S_2 l_2}{EA} + \alpha_T \Delta T\, l_2 = \frac{23098 N \cdot 1000 mm}{2{,}1 \cdot 10^5\ \frac{N}{mm^2} \cdot 314 mm^2} + 1 \cdot 10^{-5} K^{-1} \cdot 25 K \cdot 1000\ mm$$

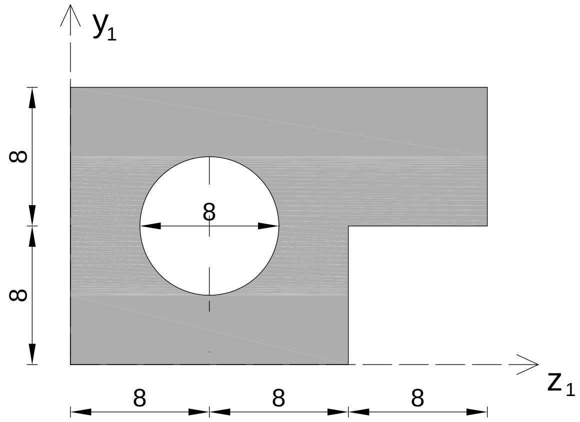
$$\Delta l_2 = 0{,}35\, mm + 0{,}25\, mm = 0{,}6\, mm$$

12. Zadatak

Za zadani poprečni presjek (čije su mjere zadane u cm) potrebno je:

a) Odrediti momente površine drugog reda (momente tromosti) I_z, I_y i I_{zy} s obzirom na težište poprečnog presjeka.

b) Odrediti glavne momente površine drugog reda (momente tromosti) poprečnog presjeka, te položaj pripadajućih glavnih osi.



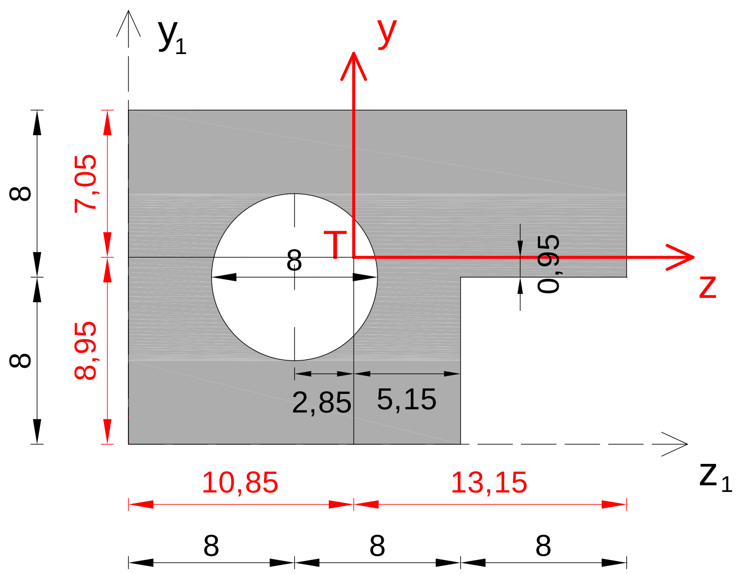
Površina poprečnog presjeka iznosi

$$A = 24 \cdot 16 - 8 \cdot 8 - 4^2 \pi = 269{,}73\ cm^2 .$$

Težište poprečnog presjeka:

$$z_T = \frac{(24 \cdot 16) \cdot 12 - (8 \cdot 8) \cdot 20 - 4^2 \pi \cdot 8}{269{,}73} = 10{,}85\ cm$$

$$y_T = \frac{(24 \cdot 16) \cdot 8 - (8 \cdot 8) \cdot 4 - 4^2 \pi \cdot 8}{269{,}73} = 8{,}95\ cm$$



Momenti površine drugog reda I_z, I_y i I_{zy} s obzirom na težište poprečnog presjeka iznose:

$$I_z = \frac{24\cdot 16^3}{12} + (24\cdot 16)\cdot 0{,}95^2 - \left[\frac{8\cdot 8^3}{12} + (8\cdot 8)\cdot 4{,}95^2 + \frac{\pi\cdot 8^4}{64} + (4^2\pi)\cdot 0{,}95^2\right] = 6382{,}6\ cm^4$$

$$I_y = \frac{16\cdot 24^3}{12} + (24\cdot 16)\cdot 1{,}15^2 - \left[\frac{8\cdot 8^3}{12} + (8\cdot 8)\cdot 9{,}15^2 + \frac{\pi\cdot 8^4}{64} + (4^2\pi)\cdot 2{,}85^2\right] = 12630{,}9\ cm^4$$

$$I_{zy} = 0 + (24\cdot 16)\cdot 1{,}15\cdot(-0{,}95) - \left[0 + (8\cdot 8)\cdot 9{,}15\cdot(-4{,}95) + 0 + (4^2\pi)\cdot(-2{,}85)\cdot(-0{,}95)\right]$$

$$I_{zy} = 2343{,}1\ cm^4$$

Budući je $I_{zy} \neq 0$, osi z i y nisu glavne osi poprečnog presjeka.

Glavni momenti površine drugog reda (momenti tromosti) poprečnog presjeka računaju se prema sljedećem izrazu:

$$I_{1,2} = \frac{I_z + I_y}{2} \pm \frac{1}{2}\sqrt{(I_z - I_y)^2 + 4I_{zy}^{\ 2}}$$

$$I_{1,2} = \frac{6382{,}6 + 12630{,}9}{2} \pm \frac{1}{2}\sqrt{(6382{,}6 - 12630{,}9)^2 + 4\cdot 2343{,}1^2}$$

$$I_{1,2} = 9506{,}75 \pm 3905{,}18$$

$$I_1 = 13411{,}93\ cm^4 \qquad I_2 = 5601{,}57\ cm^4$$

Budući je $I_z < I_y$ vrijedi:

$$I_v = I_{max} = I_1 = 13411{,}93\ cm^4$$

$$I_u = I_{min} = I_2 = 5601{,}57\ cm^4$$

Kut glavnih osi tromosti poprečnog presjeka:

$$tg\,2\varphi = -\frac{I_{zy}}{I_z - I_y} = -\frac{2\cdot 2343{,}1}{6382{,}6 - 12630{,}9} = 0{,}75$$

$$2\varphi = 36{,}87^\circ \quad \rightarrow \quad \varphi = 18{,}43^\circ$$